

SATURATION AND IRREDUNDANCY FOR $\mathrm{SPIN}(8)$

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Dedicated to F. Hirzebruch on the occasion of his seventieth birthday

ABSTRACT. We explicitly calculate the *triangle inequalities* for the group $\mathrm{PSO}(8)$, thereby explicitly solving the *eigenvalues of a sum* problem for this group (equivalently describing the side-lengths of geodesic triangles in the corresponding symmetric space for the metric d_Δ with values in the Weyl chamber Δ). We then apply some computer programs to verify two basic questions/conjectures. First, we verify that the above system of inequalities is irredundant. Then, we verify the “saturation conjecture” for the decomposition of tensor products of finite-dimensional irreducible representations of $\mathrm{Spin}(8)$. Namely, we show that for any triple of dominant weights (λ, μ, ν) such that $\lambda + \mu + \nu$ is in the root lattice, and any positive integer N ,

$$(V(\lambda) \otimes V(\mu) \otimes V(\nu))^{\mathrm{Spin}(8)} \neq 0$$

if and only if

$$(V(N\lambda) \otimes V(N\mu) \otimes V(N\nu))^{\mathrm{Spin}(8)} \neq 0.$$

1. INTRODUCTION

In this paper we address the following three basic problems in algebraic group theory. The statements of the first two problems and the description of their solution set, the cone $\mathcal{C}(R)$, do not depend on the fundamental group of G .

Let G be a connected complex semisimple algebraic group. We fix a Borel subgroup B , a maximal torus $T \subset B$ and a maximal compact subgroup K . Let $X = G/K$ be the associated symmetric space. Let $\mathfrak{b}, \mathfrak{h}, \mathfrak{k}$ and \mathfrak{g} be the Lie algebras of B, T, K and G respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. We can and will assume that \mathfrak{h} satisfies $\mathfrak{h} = \mathfrak{h} \cap \mathfrak{k} \oplus \mathfrak{h} \cap \mathfrak{p}$. Let \mathfrak{a} be the second intersection $\mathfrak{h} \cap \mathfrak{p}$ (the Cartan subspace). Let A be the real split subtorus of T corresponding to \mathfrak{a} . The choice of B determines the set $R^+ \subset \mathfrak{a}^*$ of positive roots

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and thus the set $\Pi = \{\alpha_1, \dots, \alpha_l\} \subset R^+$ of simple roots and also the fundamental weights $\{\omega_1, \dots, \omega_l\}$, l being the rank of G . The cone generated by the positive roots determines the dual cone $\Delta \subset \mathfrak{a}$, the (closed) Weyl chamber.

In Section 2 we will introduce the Δ -valued distance d_Δ on the symmetric space X . We then have the following

Problem 1. *The generalized triangle inequalities.* Give conditions on a triple $(h_1, h_2, h_3) \in \Delta^3$ that are necessary and sufficient in order that there exist a triangle in X with vertices x_1, x_2, x_3 such that $d_\Delta(x_1, x_2) = h_1$, $d_\Delta(x_2, x_3) = h_2$ and $d_\Delta(x_3, x_1) = h_3$.

Our second problem is the generalization (to general G) of the problem of finding the possible eigenvalues of a sum of Hermitian matrices given the eigenvalues of the summands. To formulate this problem, define the map

$$\pi : \mathfrak{p}/K \rightarrow \Delta$$

by taking $\pi(x)$ to be the unique point in the intersection of Δ with the $Ad K$ -orbit of x .

Problem 2. *The generalized eigenvalues of a sum problem.* Determine the subset $\mathcal{C} = \mathcal{C}(R) \subset \Delta^3$ consisting of triples $(h_1, h_2, h_3) \in \Delta^3$ such that there exists a triple $(y_1, y_2, y_3) \in \mathfrak{p}^3$ for which

$$y_1 + y_2 + y_3 = 0$$

and $\pi(y_1) = h_1, \pi(y_2) = h_2, \pi(y_3) = h_3$.

It turns out that the sets of solutions to Problems 1 and 2 coincide, see [Kl2], [AMW], [EL], [KLM1] and [KLM2]. The common solution set \mathcal{C} is in fact a convex homogeneous polyhedral cone \mathcal{C} , [BS], see also [KLM1]. The set \mathcal{C} is described in [BS] with a refinement in [KLM1] by a certain system of homogeneous linear inequalities, the *extended triangle inequalities* $\widetilde{TI}(R)$, which is, in general, a redundant system. A smaller system, the *triangle inequalities* $TI(R)$, was introduced by Belkale and Kumar in [BK]. These systems of inequalities are based on the cup product, resp. the degenerated cup product, on the cohomology of the generalized Grassmannians G/P , where $P \subset G$ are maximal parabolic subgroups.

Belkale and Kumar have posed the question if the system $TI(R)$ is irredundant (cf. [BK], Section 1.1).

Remark 1.1. *The two systems of inequalities $\widetilde{TI}(R)$ and $TI(R)$ coincide in the case of type A root systems (cf. [BK], Lemma 19). In this case irredundancy was proven by Knutson, Tao and Woodward in [KTW].*

In this paper we prove that the system $TI(R)$ is indeed irredundant for $R = D_4$, that is for groups with Lie algebra $\mathfrak{so}(8)$.

Theorem 1.2. *The system of triangle inequalities $TI(D_4)$ is irredundant.*

We will prove this by explicitly computing the system $TI(D_4)$ and then verifying that it consists of 306 inequalities, while the cone \mathcal{C} has 306 facets (and 81 extremal rays). This computation is done by applying the computer program CONVEX, [F].

Our third problem concerns the decomposition of tensor products of finite-dimensional irreducible representations of a complex semisimple group. It is natural to assume that this group is simply-connected.

In order to relate the third problem to the first two it is necessary to introduce the Langlands' dual G^\vee of G , see [Sp], pages 3-6. We explain briefly why this is the case. There is a natural correspondence of maximal tori T and T^\vee for the two groups such that the dominant coweights (the “integral points” in Δ) of G are the dominant weights of G^\vee whence *the input data for Problem 3 for the case of G^\vee is a subset consisting of the “integral points” of the input data for Problems 1 and 2 for G .*

For this reason we now assume that in the previous discussion G was the centerless form of \mathfrak{g} (since this corresponds to the assumption that the group G^\vee is simply-connected).

Let P^\vee be the weight lattice of G^\vee , i.e., P^\vee is the character lattice of the maximal torus $T^\vee \subset G^\vee$. Then,

$$\mathcal{D} = \mathcal{D}(G^\vee) := P^\vee \cap \Delta$$

is the set of dominant weights of G^\vee .

Definition 1.3. *We define $(\mathcal{D}^3)^0$ to be the subsemigroup of \mathcal{D}^3 consisting of those triples of dominant weights whose sum is in the root lattice Q^\vee of G^\vee .*

Given $\lambda \in \mathcal{D}$ let $V(\lambda)$ denote the irreducible representation of G^\vee with dominant weight λ .

We now state our third problem.

Problem 3. Determine the semigroup $\mathcal{R} = \mathcal{R}(G^\vee) \subset \mathcal{D}^3$ consisting of triples of dominant weights (λ, μ, ν) such that

$$(V(\lambda) \otimes V(\mu) \otimes V(\nu))^{G^\vee} \neq 0.$$

It is well known that Problems 2 and 3 are related, namely that the semigroup $\mathcal{C}^0 = \mathcal{C}(G^\vee)^0 := \mathcal{C}(R) \cap (\mathcal{D}^3)^0$ is the saturation of the semigroup $\mathcal{R}(G^\vee)$ in the semigroup $(\mathcal{D}^3)^0$. For a more detailed statement, see Theorem 2.5.

It was conjectured in [KM2] that \mathcal{R} is saturated in $(\mathcal{D}^3)^0$ for all the simply-laced (and simply-connected) groups G^\vee , i.e., for any $(\lambda, \mu, \nu) \in (\mathcal{D}^3)^0$ and any positive integer N , if $(N\lambda, N\mu, N\nu) \in \mathcal{R}$, then $(\lambda, \mu, \nu) \in \mathcal{R}$, i.e., $\mathcal{R}(G^\vee) = \mathcal{C}(G^\vee)^0$. This conjecture is again known in the case of type A root systems, this is the *saturation theorem* of Knutson and Tao [KT], see also [B, DW, KM1] for alternative proofs.

We now state our second main theorem.

Theorem 1.4. *Let $R = D_4$ so that $G^\vee = Spin(8)$. A triple $(\lambda, \mu, \nu) \in (\mathcal{D}^3)^0$ satisfies $(\lambda, \mu, \nu) \in \mathcal{R}(Spin(8))$ if and only if $(\lambda, \mu, \nu) \in \mathcal{C}(D_4)$. Equivalently, the semigroup $\mathcal{R}(Spin(8))$ is saturated in the semigroup $(\mathcal{D}^3)^0$.*

In order to prove Theorem 1.4 we use the computer program 4ti2, [HHM], to compute the Hilbert basis of the semigroup $\mathcal{C}(Spin(8))^0$. It turns out that this basis consists of 82 elements (just one more than the number of extremal rays). Moreover, modulo the permutations of the vectors λ, μ, ν and action of the automorphisms of the Dynkin diagram of D_4 , there are only 10 different semigroup generators. For each of these generator $(\lambda_i, \mu_i, \nu_i)$ we verify that

$$(\lambda_i, \mu_i, \nu_i) \in \mathcal{R}$$

by applying the MAPLE package WEYL, [S]. Since \mathcal{R} is a semigroup, it then follows that $\mathcal{C}^0 = \mathcal{R}$.

By [KLM3], Section 9.4, the previous theorem implies the following saturation theorem for the structure constants of the spherical Hecke algebra of $PSO(8)$. Considering $PSO(8)$ as a group scheme $\underline{PSO(8)}$ over \mathbb{Z} , let \mathcal{G} be the group of its rational points in a nonarchimedean local field \mathbb{K} . Let \mathcal{O} be the ring of integers (elements of nonnegative valuation) of \mathbb{K} . We let \mathcal{K} be the group of \mathcal{O} -rational points of $PSO(8)$. Let $\mathcal{H}_{\mathcal{G}}$ denote the associated spherical Hecke ring. We recall that the set of dominant coweights \mathcal{D} of G parametrizes the \mathcal{K} -double cosets in \mathcal{G} and that the ring $\mathcal{H}_{\mathcal{G}}$ is free over \mathbb{Z} with basis the characteristic functions $\{f_\lambda : \lambda \in \mathcal{D}\}$. We let $*$ denote the (convolution) product in $\mathcal{H}_{\mathcal{G}}$. We have

Theorem 1.5. *Let $G = PSO(8)$. For $\lambda, \mu, \nu \in \mathcal{D}$, the characteristic function of the identity \mathcal{K} -double coset occurs in the expansion of the product $f_\lambda * f_\mu * f_\nu$ in terms of the above basis if and only if the triple $(\lambda, \mu, \nu) \in \mathcal{C}$ and $\lambda + \mu + \nu$ is in the coroot lattice Q^\vee of G .*

2. FURTHER DISCUSSION OF THE THREE PROBLEMS

In this section we give some more details about the three problems formulated in the Introduction. We follow the same notation (as in the Introduction). In particular, G is a complex semisimple adjoint group (with root system R) and G^\vee is its Langlands' dual, which is simply-connected (since G is adjoint).

2.1. The distance d_Δ . We now define the Δ -valued distance d_Δ . Let A_Δ be the image of Δ under the exponential map $\exp : \mathfrak{g} \rightarrow G$. We will need the following basic theorem, the Cartan decomposition for the group G , see [He], Theorem 1.1, pg. 402.

Theorem 2.1. *We have*

$$G = KA_\Delta K.$$

Moreover, for any $g \in G$, the intersection of the double coset KgK with A_Δ consists of a single point to be denoted $a(g)$.

Let $\overline{x_1 x_2}$ be the oriented geodesic segment in $X = G/K$ joining the point x_1 to the point x_2 . Then there exists an element $g \in G$ which sends x_1 to the base point $o = eK$ and x_2 to $y = \exp(\delta)$ where $\delta \in \Delta$. Note that the point δ is uniquely determined by $\overline{x_1 x_2}$. We define a map σ from G -orbits of oriented geodesic segments to Δ by

$$\sigma(\overline{x_1 x_2}) = \delta.$$

Clearly we have the following consequence of the Cartan decomposition.

Lemma 2.2. *The map σ gives rise to a one-to-one correspondence between the G -orbits of oriented geodesic segments in X and the points of Δ .*

In the rank 1 case $\sigma(\overline{x_1 x_2})$ is just the length of the geodesic segment $\overline{x_1 x_2}$.

Definition 2.3. *We call $\sigma(\overline{x_1 x_2})$ the Δ -length of $\overline{x_1 x_2}$ or the Δ -distance between x_1 and x_2 . We write $d_\Delta(x_1, x_2) = \sigma(\overline{x_1 x_2})$.*

We note the formula

$$d_\Delta(x_1, x_2) = \log a(g_1^{-1} g_2) \text{ where } x_1 = g_1 K, x_2 = g_2 K.$$

Remark 2.4. *The delta distance is symmetric in the sense that*

$$d_\Delta(x_1, x_2) = -w_o d_\Delta(x_2, x_1),$$

where w_o is the unique longest element in the Weyl group.

2.2. The relations between Problems 1, 2 and 3. In this subsection we expand the discussion in the Introduction concerning the relations between the three problems. We first discuss the relation between Problems 2 and 3.

The (a)-part of the following theorem is standard, see for example the appendix of [KLM3]. The (b)-part follows from Theorem 1.2 of [KLM1], see also Theorem 1.3 of [KLM2] and the paragraph following it. Of course, the (b)-part is clear for the simply-laced groups. So, the only nontrivial case is essentially that of the group G corresponding to the root systems of type B_l . In this case, Belkale and Kumar have shown that the triangle inequalities themselves coincide under the identification of \mathfrak{a} with \mathfrak{a}^* (via any Killing form).

Theorem 2.5. *(a) For any semisimple adjoint group G with root system R , under the identification of \mathfrak{a} with \mathfrak{a}^* (via any Killing form),*

$$\mathcal{R}(G^\vee) \subset \mathcal{C}(R^\vee).$$

Conversely, for any triple $(\lambda, \mu, \nu) \in \mathcal{C}(R^\vee) \cap \mathcal{D}^3$, there exists a positive integer N such that $(N\lambda, N\mu, N\nu) \in \mathcal{R}(G^\vee)$.

(b) Under the identification of \mathfrak{a}^ with \mathfrak{a} ,*

$$\mathcal{C}(R) = \mathcal{C}(R^\vee).$$

Thus, combining the two parts, we get the following intrinsic inclusion:

$$\mathcal{R}(G^\vee) \subset \mathcal{C}(R).$$

We recall the following standard definition. Suppose that $S_1 \subset S_2$ is an inclusion of semigroups. Then the *saturation* of S_1 in S_2 is the semigroup of elements $x \in S_2$ such that there exists $n \in \mathbb{Z}_+$ with $nx \in S_1$.

Remark 2.6. *We may restate the previous theorem by saying that the semigroup $\mathcal{C}(G^\vee)^0 := \mathcal{C}(R) \cap (\mathcal{D}^3)^0$ is the saturation of the semigroup $\mathcal{R}(G^\vee)$ in the semigroup $(\mathcal{D}^3)^0$.*

We conclude this section by briefly indicating why the solutions of Problems 1 and 2 coincide. First of all, Problem 2 (in the case of an n -fold sum) can be reformulated geometrically as a problem of the existence of geodesic polygons in \mathfrak{p} with fixed Δ -valued side-lengths. Both \mathfrak{p} and the symmetric space X admit compactifications by a “visual” sphere S which also has the structure of a spherical building $\partial_{\text{Tits}} X$. The vertices of this building are points in the flag manifolds G/P (where P ’s are the maximal parabolic subgroups in G).

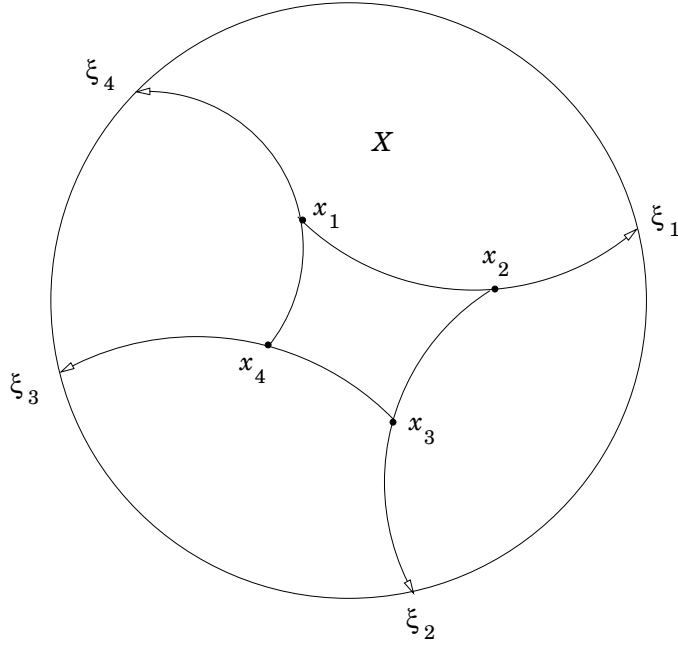


FIGURE 1. Gauss map.

Then one can define the *Gauss map* Γ which sends the geodesic polygon $[x_1, \dots, x_n] \subset X$ to the weighted configuration

$$\Gamma([x_1, \dots, x_n]) = ((m_1, \xi_1), \dots, (m_n, \xi_n))$$

of points in S . Here $m_i := d(x_i, x_{i+1})$ are the ordinary distances, which serve as *weights* at the points $\xi_i \in S$. The same definition also works for X replaced by \mathfrak{p} . The Gauss map from quadrilaterals in the hyperbolic plane to configurations of four points $\xi_1, \xi_2, \xi_3, \xi_4$ on the visual boundary (the circle) is depicted in figure 1. The key problem then is to identify the images of Gauss maps Γ for X and \mathfrak{p} . It turns out that both consist of nice semistable weighted configurations on S , where the notion of stability is essentially the one introduced by Mumford in Geometric Invariant Theory (in the case when the weights are natural numbers). We refer the reader to Theorems 5.2 and 5.9 of [KLM1] for the precise statements.

Therefore, Problems 1 and 2 are both equivalent to the existence problem for semistable weighted configurations on S and hence Problems 1 and 2 are equivalent.

3. THE TRIANGLE INEQUALITIES

We need more notation. We let $S = \{s_1, \dots, s_l\}$ be the set of (simple) reflections in the root hyperplanes defined by the simple roots and let $W \subset \text{Aut } \mathfrak{a}$ be the Weyl group generated by S .

Let $\{x_i\}$ be the basis of \mathfrak{h} dual to the basis Π , i.e., $\alpha_i(x_j) = \delta_{i,j}$. We let ℓ be the length function on W . Let α_i^\vee be the coroot corresponding to the root α_i . Also, for a standard parabolic subgroup P of G (i.e. $P \supset B$), we let W_P be the subgroup of elements with representatives in P and W^P denote the set of shortest length representatives for the cosets W/W_P (we recall that each coset has a unique shortest length representative). Let w_o^P be the unique longest element in W^P .

3.1. The extended triangle inequalities. We now describe the solution of Problem 1 of the Introduction, that is the description of the inequalities determining the Δ -valued side-lengths of geodesic triangles in X .

3.1.1. The weak triangle inequalities. We first describe a natural subsystem of the triangle inequalities. The naive triangle inequality

$$d_\Delta(x_1, x_3) \leq_\Delta d_\Delta(x_1, x_2) + d_\Delta(x_2, x_3)$$

does not hold [KLM3]. Here the order \leq_Δ is the one defined by the (acute) cone Δ . This can be remedied if we replace Δ by the dual (obtuse) cone Δ^* and let \leq_{Δ^*} denote the associated order. Then, the analogue of the above inequality holds and, in fact, for any element w of the Weyl group W , the inequality

$$w \cdot d_\Delta(x_1, x_3) \leq_{\Delta^*} w \cdot d_\Delta(x_1, x_2) + d_\Delta(x_2, x_3)$$

holds. We call the resulting system of inequalities (as w varies) the *weak triangle inequalities* to be denoted $WTI(R)$.

For the root systems R of ranks one and two, the weak triangle inequalities already give a solution to Problems 1 and 2 of the Introduction. However, they are no longer sufficient in ranks three or more.

3.1.2. The extended triangle inequalities. We now describe a system of linear inequalities on \mathfrak{a}^3 which describes the cone $\mathcal{C}(R)$. However, this system is usually not irredundant. These inequalities (based on the cup-product of Schubert classes) will be called the *extended triangle inequalities*. The system of extended triangle inequalities is independent of the choice of G corresponding to a fixed Lie algebra \mathfrak{g} , hence depends only on the root system R associated to G . We denote the system of extended triangle inequalities by $\widehat{TI}(R)$ or just \widehat{TI} when the reference to R is clear.

As a consequence of the Bruhat decomposition,

$$G = \bigsqcup_{w \in W^P} BwP,$$

the generalized flag variety G/P is the disjoint union of the subsets

$$\{C_w^P := BwP/P\}_{w \in W^P}.$$

The subset C_w^P is biregular isomorphic to the affine space $\mathbb{C}^{\ell(w)}$ and is called a *Schubert cell*, where $\ell(w)$ is the length of w . The closure X_w^P of C_w^P is called a *Schubert variety*. We will use $[X_w^P]$ to denote the integral homology class in $H_*(G/P)$ carried by X_w^P . Then, the integral homology $H_*(G/P)$ is a free \mathbb{Z} -module with basis $\{[X_w^P] : w \in W^P\}$.

Let $\{\epsilon_w^P : w \in W^P\}$ denote the dual basis of $H^*(G/P)$ under the Kronecker pairing $\langle \cdot, \cdot \rangle$ between homology and cohomology. Thus, we have for $w, w' \in W^P$,

$$\langle \epsilon_w^P, [X_{w'}^P] \rangle = \delta_{w, w'}.$$

The system of extended triangle inequalities breaks up into $\text{rank}(\mathfrak{g})$ subsystems \widetilde{TI}^P , where P runs over standard maximal parabolic subgroups. The subsystem \widetilde{TI}^P is controlled by the Schubert calculus in the generalized Grassmannian G/P in the sense that there is one inequality $T_{\mathbf{w}}^P$ for each triple of elements $\mathbf{w} = (w_1, w_2, w_3) \in W^P$ such that

$$\epsilon_{w_1}^P \cdot \epsilon_{w_2}^P \cdot \epsilon_{w_3}^P = \epsilon_{w_0^P}^P$$

in $H^*(G/P)$. To describe the inequality $T_{\mathbf{w}}^P$, let λ be the fundamental weight corresponding to P . Then the action of W on \mathfrak{a}^* induces a one-to-one correspondence $f : W^P \rightarrow W\lambda$. Thus, we may reparameterize the Schubert classes in G/P by elements of $W\lambda \subset \mathfrak{a}^*$. We let $\lambda_i = f(w_i)$, $i = 1, 2, 3$. Then the inequality $T_{\mathbf{w}}^P$ is given by

$$\lambda_1(h_1) + \lambda_2(h_2) + \lambda_3(h_3) \geq 0, \quad (h_1, h_2, h_3) \in \Delta^3.$$

3.2. The triangle inequalities. As we have mentioned earlier, the system of extended triangle inequalities is in general not an irredundant system. We now describe the subsystem of triangle inequalities.

To this end we recall the definition of the new product \odot_0 in the cohomology $H^*(G/P)$ introduced by Belkale-Kumar [BK, Sect. 6]. We only need to consider the case when P is a standard maximal parabolic subgroup. In this case, we set $x_P = x_{i_P}$, where s_{i_P} is the unique simple reflection not in W_P . Similarly, we set $\omega_P = \omega_{i_P}$. We can identify W^P with the orbit $W \cdot \omega_P$. For $w \in W^P$, let $\lambda_w = \lambda_w^P$ denote $w(\omega_P)$; this is called the *maximally singular weight* corresponding to w .

Write the cup product in $H^*(G/P)$ as follows:

$$\epsilon_u^P \cdot \epsilon_v^P = \sum_{w \in W^P} d_{u,v}^w \epsilon_w^P.$$

Then, by definition,

$$\epsilon_u^P \odot_0 \epsilon_v^P = \sum_{w \in W^P} d_{u,v}^w \delta_{u,v}^w \epsilon_w^P,$$

where $\delta_{u,v}^w := 1$ if $(u^{-1}\rho + v^{-1}\rho - w^{-1}\rho - \rho)(x_P) = 0$ and $\delta_{u,v}^w := 0$ otherwise, where ρ is the (standard) half sum of positive roots of \mathfrak{g} .

Recall that $\pi : \mathfrak{p}/K \rightarrow \Delta$ is defined by intersecting an AdK -orbit with Δ . Then [BK, Theorem 28] gives the following solution of Problem 2 stated in the Introduction:

Theorem 3.1. *Let $(h_1, \dots, h_n) \in \Delta^n$. Then, the following are equivalent:*

(a) *There exists $(y_1, \dots, y_n) \in \mathfrak{p}^n$ such that $\sum_{j=1}^n y_j = 0$ and $\pi(y_j) = h_j$ for all $j = 1, \dots, n$.*

(b) *For every standard maximal parabolic subgroup P in G and every choice of n -tuple $\mathbf{w} = (w_1, \dots, w_n) \in (W^P)^n$ such that*

$$\epsilon_{w_1}^P \odot_0 \cdots \odot_0 \epsilon_{w_n}^P = \epsilon_{w_P}^P \in (H^*(G/P), \odot_0),$$

the following inequality holds:

$$(T_{\mathbf{w}}^P) \quad \sum_{j=1}^n \lambda_{w_j}^P(h_j) \geq 0.$$

The collection of inequalities $\{T_{\mathbf{w}}^P\}$, such that \mathbf{w} and P are as in (b), is called the *triangle inequalities*.

Remark 3.2. *As was the case for $n = 3$, the statement in (a) is equivalent to the existence of a geodesic n -gon in X with d_Δ -side-lengths h_1, h_2, \dots, h_n .*

4. DETERMINATION OF THE PRODUCT \odot_0 IN $H^*(G/P)$

From now on, the group G will be taken to be the adjoint group of type D_4 , i.e., $G = PSO(8)$. Since G is simply-laced, the Langlands' dual G^\vee has the same root system as G . Moreover, G being the adjoint group, G^\vee is the simply-connected cover of G , i.e., $G^\vee = Spin(8)$. We will only need to consider the maximal parabolic subgroups. We will abbreviate the classes ϵ_w^P for $w \in W^P$ by b_i^j according to the following tables. Here the subscript i denotes half of the cohomological degree of b_i^j , i.e., $b_i^j \in H^{2i}(G/P)$, and j runs over the indexing set with cardinality

equal to the rank of $H^{2i}(G/P)$. In the case that $H^{2i}(G/P)$ is of rank one, we suppress the superscript j . Moreover, in the following tables, we also list the maximally singular weight $\lambda_w := w\omega_P$ associated to the element $w \in W^P$ as well as the value $n_w := (w^{-1}\rho)(x_P)$. We express λ_w in terms of the standard coordinates $\{\epsilon_i\}_{i=1,\dots,4}$ of \mathfrak{h}^* as given in [Bo, Planche IV]. We follow the following indexing convention as in loc cit.

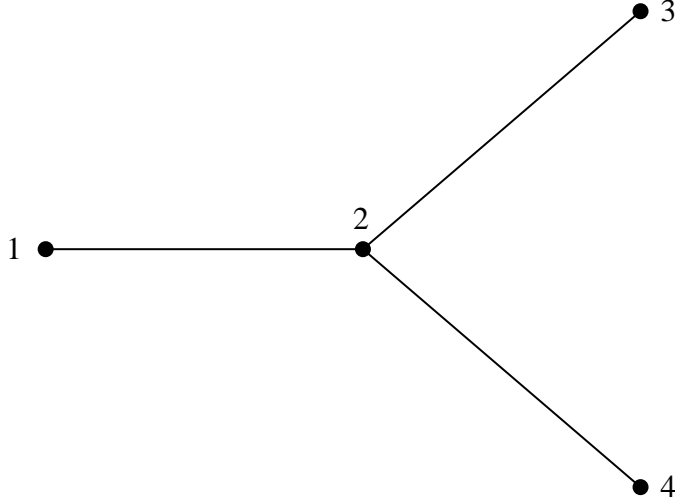


FIGURE 2. *Dynkin diagram for D_4 .*

For any $1 \leq i \leq 4$, let P_i be the maximal parabolic subgroup of G corresponding to the i -th node of the Dynkin diagram, i.e., W_{P_i} is generated by the simple reflections $\{s_j; j \neq i\}$.

To determine the cohomology $H^*(G/P_i)$ under the product \odot_0 , for all the maximal parabolic subgroups P_i , because of the Dynkin automorphisms, we only need to determine it for $i = 1, 2$. Moreover, since P_1 is a minuscule parabolic in $\text{Spin}(8)$, the product \odot_0 coincides with the cup product by [BK, Lemma 19].

In what follows we will use the symbol F to denote the group of automorphisms of the Dynkin diagram of D_4 , so F is isomorphic to the symmetric group S_3 on the labels $\{1, 3, 4\}$ of the Dynkin diagram.

4.1. Determination of $(H^*(G/P_1), \odot_0)$. The longest element w_o of W is given by

$$(1) \quad w_o = s_4 s_2 s_1 s_4 s_2 s_4 s_3 s_2 s_4 s_1 s_2 s_3,$$

and it is central in W . Moreover, the longest element w_{o,P_1} of W_{P_1} is given by

$$(2) \quad w_{o,P_1} = s_3 s_2 s_4 s_3 s_2 s_3.$$

Thus, the longest element $w_o^{P_1}$ of W^{P_1} is given by (cf. [KuLM, Proposition 2.6])

$$(3) \quad w_o^{P_1} = w_o w_{o,P_1} = s_1 s_2 s_3 s_4 s_2 s_1.$$

From this and the fact that $|W^{P_1}| = 8$, we see that the elements of W^{P_1} are enumerated as in the chart below. To calculate n_w , use the general formula (cf. [K, Corollary 1.3.22]) for any $w \in W$:

$$(4) \quad \rho - w^{-1}\rho = \sum_{R^+ \cap w^{-1}R^-} \alpha,$$

and for any parabolic subgroup P of G and any $w \in W^P$,

$$(5) \quad R^+ \cap w^{-1}R^- \subset R^+ \setminus R_P^+,$$

where $R^- := -R^+$ and R_P^+ is the set of positive roots in the Levi component of P . Since P_1 is a minuscule maximal parabolic subgroup, for any $w \in W^{P_1}$, by (4) and (5) we get

$$(6) \quad (\rho - w^{-1}\rho)(x_{P_1}) = \ell(w).$$

From this, the value of n_w given in the following chart can easily be verified since $\rho(x_{P_1}) = 3$. The value of λ_w is obtained by explicit calculations.

$\epsilon_w^{P_1}$	w	λ_w	n_w
$b_0 = 1$	e	$(1, 0, 0, 0)$	3
b_1	s_1	$(0, 1, 0, 0)$	2
b_2	$s_2 s_1$	$(0, 0, 1, 0)$	1
b_3^1	$s_3 s_2 s_1$	$(0, 0, 0, 1)$	0
b_3^2	$s_4 s_2 s_1$	$(0, 0, 0, -1)$	0
b_4	$s_3 s_4 s_2 s_1$	$(0, 0, -1, 0)$	-1
b_5	$s_2 s_3 s_4 s_2 s_1$	$(0, -1, 0, 0)$	-2
b_6	$s_1 s_2 s_3 s_4 s_2 s_1$	$(-1, 0, 0, 0)$	-3

Using [KuLM, Lemma 2.9] and the Chevalley formula (cf. [K, Theorem 11.1.7(i)]), all the products in the following table can be determined except the products of b_2 with b_2 and b_3^* . Since $b_1 b_1 = b_2$, using the Chevalley formula twice, we get these products as well.

Multiplication table for G/P_1 under the product \odot_0 :

\odot_0	b_1	b_2	b_3^1	b_3^2
b_1	b_2			
b_2	$b_3^1 + b_3^2$	$2b_4$		
b_3^1	b_4	b_5	0	
b_3^2	b_4	b_5	b_6	0
b_4	b_5	b_6	0	0
b_5	b_6	0	0	0
b_6	0	0	0	0

4.2. Determination of $(H^*(G/P_2), \odot_0)$. For any parabolic subgroup P , let θ^P be the involution of W^P defined by

$$(7) \quad \theta^P w = w_o w w_{o,P}.$$

Then, by [KuLM, Section 2.1], ϵ_w^P is Poincaré dual to $\epsilon_{\theta^P w}^P$.

Using (1) and

$$(8) \quad w_{o,P_2} = s_1 s_3 s_4,$$

we get

$$(9) \quad w_o^{P_2} = s_2 s_4 s_1 s_2 s_3 s_2 s_1 s_4 s_2.$$

The enumeration of W^{P_2} as in the following table can be read off from (9) together with the fact that $|W^{P_2}| = 24$. The values of λ_w and n_w are obtained by explicit calculations. Observe that the following identities provide some simplification in the calculations of λ_w and n_w :

For any $w \in W^{P_2}$,

$$(10) \quad \lambda_{\theta^P(w)} = -\lambda_w,$$

and

$$(11) \quad n_{\theta^P(w)} = -n_w.$$

In the following table, the two w 's appearing in the same row are θ^P -images of each other, i.e., the corresponding classes $\epsilon_w^{P_2}$ are Poincaré dual to each other.

$\epsilon_w^{P_2}$	w	λ_w	n_w	$\epsilon_w^{P_2}$	w	λ_w	n_w
$b_0 = 1$	e	$(1, 1, 0, 0)$	5	b_9	$s_2 s_4 s_1 s_2 s_3 s_2 s_1 s_4 s_2$	$(-1, -1, 0, 0)$	-5
b_1	s_2	$(1, 0, 1, 0)$	4	b_8	$s_4 s_1 s_2 s_3 s_2 s_1 s_4 s_2$	$(-1, 0, -1, 0)$	-4
b_2^1	$s_1 s_2$	$(0, 1, 1, 0)$	3	b_7^1	$s_4 s_2 s_3 s_2 s_1 s_4 s_2$	$(0, -1, -1, 0)$	-3
b_2^2	$s_3 s_2$	$(1, 0, 0, 1)$	3	b_7^2	$s_4 s_2 s_1 s_2 s_3 s_4 s_2$	$(-1, 0, 0, -1)$	-3
b_2^3	$s_4 s_2$	$(1, 0, 0, -1)$	3	b_7^3	$s_3 s_2 s_1 s_2 s_4 s_3 s_2$	$(-1, 0, 0, 1)$	-3
b_3^1	$s_3 s_1 s_2$	$(0, 1, 0, 1)$	2	b_6^1	$s_4 s_2 s_3 s_1 s_4 s_2$	$(0, -1, 0, -1)$	-2
b_3^2	$s_4 s_1 s_2$	$(0, 1, 0, -1)$	2	b_6^2	$s_3 s_2 s_4 s_1 s_3 s_2$	$(0, -1, 0, 1)$	-2
b_3^3	$s_4 s_3 s_2$	$(1, 0, -1, 0)$	2	b_6^3	$s_1 s_2 s_3 s_4 s_1 s_2$	$(-1, 0, 1, 0)$	-2
b_4^1	$s_2 s_3 s_1 s_2$	$(0, 0, 1, 1)$	1	b_5^1	$s_4 s_2 s_1 s_3 s_2$	$(0, 0, -1, -1)$	-1
b_4^2	$s_4 s_3 s_1 s_2$	$(0, 1, -1, 0)$	1	b_5^2	$s_2 s_4 s_1 s_3 s_2$	$(0, -1, 1, 0)$	-1
b_4^3	$s_2 s_4 s_1 s_2$	$(0, 0, 1, -1)$	1	b_5^3	$s_3 s_2 s_1 s_4 s_2$	$(0, 0, -1, 1)$	-1
b_4^4	$s_2 s_4 s_3 s_2$	$(1, -1, 0, 0)$	1	b_5^4	$s_1 s_2 s_3 s_4 s_2$	$(-1, 1, 0, 0)$	-1

From the definition of \odot_0 and the values of n_w , we get the following.

Corollary 1. *For $u, v \in W^{P_2}$, in $(H^*(G/P_2), \odot_0)$,*

$$\begin{aligned}
\epsilon_u^{P_2} \odot_0 \epsilon_v^{P_2} &= \epsilon_u^{P_2} \cdot \epsilon_v^{P_2}, \text{ if } \ell(u) + \ell(v) \leq 4 \\
&= \epsilon_u^{P_2} \cdot \epsilon_v^{P_2}, \text{ if } \ell(u) + \ell(v) \geq 5 \text{ and one of } \ell(u) \text{ or } \ell(v) \geq 5 \\
&= 0, \text{ if } \ell(u) + \ell(v) \geq 5 \text{ and both of } \ell(u) \text{ and } \ell(v) \leq 4.
\end{aligned}$$

For any $i \neq j \in \{1, 3, 4\}$, let $\sigma_{i,j}$ be the involution of $H^*(G/P_2)$ induced from the Dynkin diagram involution taking the i -th node to the j -th node and fixing the other two nodes. Let \hat{F} be the group of automorphisms of $H^*(G/P_2)$ generated by $\sigma_{1,3}, \sigma_{1,4}$ and $\sigma_{3,4}$. Then, \hat{F} is isomorphic with the symmetric group S_3 .

Using [KuLM, Lemma 2.9], the Chevalley formula and Corollary 1, we only need to calculate $b_2 b_2^*, b_2 b_5^*, b_2 b_6^*$ and $b_3 b_5^*$. Further, using the automorphism group \hat{F} , it suffices to calculate $b_2^1 b_2^*, b_2^1 b_5^*, b_2^1 b_6^*$ and $b_3^1 b_5^*$. To calculate $b_2^1 b_6^*$, write

$$b_2^1 b_6^* = d b_8, \text{ for some } d.$$

Multiply this equation by b_1 and use the known part of the multiplication table to determine d . The calculation of $b_3^1 b_5^*$ is exactly similar.

To calculate $b_2^1 b_5^*$, write

$$b_2^1 b_5^* = \sum_{i=1}^3 d_i b_7^i, \text{ for some } d_i \in \mathbb{Z}_+.$$

Multiplying the above equation by b_1 , we get

$$b_2^1 b_5^* b_1 = \sum_{i=1}^3 d_i b_8.$$

On the other hand,

$$b_2^1 b_5^2 b_1 = \sum_{i=1}^3 b_2^1 b_6^i = 2b_8.$$

Thus, $d_1 + d_2 + d_3 = 2$. Using the involution $\sigma_{3,4}$ of $H^*(G/P_2)$, we are forced to have

$$b_2^1 b_5^2 = b_7^2 + b_7^3.$$

The calculation for $b_2^1 b_5^4$ is similar. To calculate $b_2^1 b_5^1$, write

$$b_1 b_1 b_5^1 = \sum_{i=1}^3 b_2^i b_5^1.$$

But,

$$b_2^3 b_5^1 = \sigma_{1,4}(b_2^1 b_5^4) = 0,$$

and $b_2^2 b_5^1 = \sigma_{1,3}(b_2^1 b_5^1)$. On the other hand

$$b_1 b_1 b_5^1 = b_1 b_6^1 = b_7^1 + b_7^2.$$

Thus,

$$b_2^1 b_5^1 = b_7^1 \text{ or } b_7^2.$$

If $b_2^1 b_5^1 = b_7^2$, then

$$b_2^1 b_5^3 = \sigma_{3,4}(b_2^1 b_5^1) = b_7^3.$$

From this we conclude that $b_2^1 b_2^1 = 0$. However, by considering the morphism $P_4/B \rightarrow \text{Spin}(8)/B$, induced from the inclusion, we can easily see that $b_2^1 b_2^1 \neq 0$. This contradiction forces $b_2^1 b_5^1 = b_7^1$. Using $\sigma_{3,4}$ as above, we can calculate $b_2^1 b_5^3$ from $b_2^1 b_5^1$.

To calculate $b_2^1 b_2^*$, write

$$b_2^1 b_2^* = \sum_{i=1}^4 d_i b_4^i, \text{ for some } d_i \in \mathbb{Z}_+.$$

Multiply this equation by b_5^i to get

$$b_2^1 b_2^* b_5^i = d_i.$$

Now, from the known part of the multiplication table, the d_i can be determined.

Multiplication table for G/P_2 under the product \odot_0 :

\odot_0	b_1	b_2^1	b_2^2	b_2^3	b_3^1	b_3^2	b_3^3	b_4^1	b_4^2	b_4^3	b_4^4
b_1	$b_2^1 + b_2^2 + b_2^3$										
b_2^1	$b_3^1 + b_3^2$	$b_4^1 + b_4^3$									
b_2^2	$b_3^1 + b_3^3$	b_4^2	$b_4^1 + b_4^4$								
b_2^3	$b_3^2 + b_3^3$	b_4^2	b_4^2	$b_4^3 + b_4^4$							
b_3^1	$b_4^1 + b_4^2$	0	0	0	0						
b_3^2	$b_4^2 + b_4^3$	0	0	0	0	0					
b_3^3	$b_4^2 + b_4^4$	0	0	0	0	0	0				
b_4^1	0	0	0	0	0	0	0	0			
b_4^2	0	0	0	0	0	0	0	0	0		
b_4^3	0	0	0	0	0	0	0	0	0	0	
b_4^4	0	0	0	0	0	0	0	0	0	0	0
b_5^1	b_6^1	b_7^1	b_7^2	0	b_8	0	0	b_9	0	0	0
b_5^2	$b_6^1 + b_6^2 + b_6^3$	$b_7^2 + b_7^3$	$b_7^1 + b_7^3$	$b_7^1 + b_7^2$	b_8	b_8	b_8	0	b_9	0	0
b_5^3	b_6^2	b_7^1	0	b_7^3	0	b_8	0	0	0	b_9	0
b_5^4	b_6^3	0	b_7^2	b_7^3	0	0	b_8	0	0	0	b_9
b_6^1	$b_7^1 + b_7^2$	b_8	b_8	0	b_9	0	0	0	0	0	0
b_6^2	$b_7^1 + b_7^3$	b_8	0	b_8	0	b_9	0	0	0	0	0
b_6^3	$b_7^2 + b_7^3$	0	b_8	b_8	0	0	b_9	0	0	0	0
b_7^1	b_8	b_9	0	0	0	0	0	0	0	0	0
b_7^2	b_8	0	b_9	0	0	0	0	0	0	0	0
b_7^3	b_8	0	0	b_9	0	0	0	0	0	0	0
b_8	b_9	0	0	0	0	0	0	0	0	0	0

5. THE TRIANGLE INEQUALITIES FOR D_4

Consider the basis $\{\epsilon_i^*\}_{i=1,\dots,4}$ of \mathfrak{h} which is dual to the standard basis $\{\epsilon_i\}_{i=1,\dots,4}$ of \mathfrak{h}^* as in [Bo], Planche IV. Express any $h \in \mathfrak{a}$ in this basis:

$$h = x\epsilon_1^* + y\epsilon_2^* + z\epsilon_3^* + w\epsilon_4^*, \quad x, y, z, w \in \mathbb{R}.$$

Then,

$$h \in \Delta \text{ iff } x \geq y \geq z \geq |w|.$$

5.1. The system of inequalities corresponding to the parabolic subgroup P_1 . We give below the complete list (up to a permutation) of the Schubert classes $(b_{i_1}^{j_1}, b_{i_2}^{j_2}, b_{i_3}^{j_3}) = (\epsilon_{w_1}^{P_1}, \epsilon_{w_2}^{P_1}, \epsilon_{w_3}^{P_1})$ such that

$$\epsilon_{w_1}^{P_1} \odot_0 \epsilon_{w_2}^{P_1} \odot_0 \epsilon_{w_3}^{P_1} = \epsilon_{w_o}^{P_1}$$

and write down the corresponding inequality $T_{\mathbf{w}}^{P_1}$:

$$\sum_{j=1}^3 \langle \lambda_{w_j}^{P_1}, h_j \rangle \geq 0.$$

We express $h_j = (x_j, y_j, z_j, w_j)$, $j = 1, 2, 3$ in the coordinates $\{\epsilon_i^*\}_{i=1,\dots,4}$. We divide the set of inequalities in two disjoint sets, one coming from

the Schubert classes $(b_{i_1}^{j_1}, b_{i_2}^{j_2}, b_{i_3}^{j_3})$ such that at least one of the cohomology classes is 1. It can be seen that the corresponding inequalities are the *weak triangle inequalities WTI* defined in Section 3.1.1. The remaining inequalities are called the *essential triangle inequalities ETI*. We label the inequalities *ETI* corresponding to the parabolic P_1 by *ETI(1)* and similarly for *WTI*.

ETI(1):

$$(b_1, b_1, b_4) : \quad y_1 + y_2 - z_3 \geq 0 \quad (3)$$

$$(b_1, b_2, b_3^1) : \quad y_1 + z_2 + w_3 \geq 0 \quad (6)$$

$$(b_2, b_3^2, b_1) : \quad z_1 - w_2 + y_3 \geq 0 \quad (6)$$

WTI(1):

$$(1, 1, b_6) : \quad x_1 + x_2 - x_3 \geq 0 \quad (3)$$

$$(1, b_1, b_5) : \quad x_1 + y_2 - y_3 \geq 0 \quad (6)$$

$$(1, b_2, b_4) : \quad x_1 + z_2 - z_3 \geq 0 \quad (6)$$

$$(1, b_3^1, b_3^2) : \quad x_1 + w_2 - w_3 \geq 0 \quad (6)$$

To get the full set of inequalities $T_{\mathbf{w}}^{P_1}$ for P_1 , we need to permute the above collection of inequalities where the subscripts $\{1, 2, 3\}$ are permuted arbitrarily. The number at the end of each inequality denotes the number of inequalities obtained by permuting that particular inequality. Thus, the total number of inequalities $T_{\mathbf{w}}^{P_1}$ corresponding to P_1 is 36.

5.2. The system of inequalities $T_{\mathbf{w}}^{P_2}$ corresponding to the parabolic subgroup P_2 . In each cohomological degree except for 8 and 10 there is only one orbit of the Schubert classes under \hat{F} . In degree 8 there are two orbits (of three classes in the orbit of b_4^1 and one in the orbit of b_4^2).

Of course, \hat{F} acts diagonally on the set of triples $(b_{i_1}^{j_1}, b_{i_2}^{j_2}, b_{i_3}^{j_3})$ such that $b_{i_1}^{j_1} \odot_0 b_{i_2}^{j_2} \odot_0 b_{i_3}^{j_3} = b_9$. Also, S_3 acts on such triples via permutation and these two actions commute. So, we get an action of the product group $S_3 \times \hat{F}$ on such triples. The following is a complete list of such triples of Schubert classes in $H^*(G/P_2)$ up to the action of $S_3 \times \hat{F}$ and the corresponding inequality $T_{\mathbf{w}}^{P_2}$. The number at the end of each inequality denotes the order of the corresponding $S_3 \times \hat{F}$ -orbit.

ETI(2):

$$\begin{aligned}
(b_1, b_1, b_7^1) : \quad & x_1 + z_1 + x_2 + z_2 \geq y_3 + z_3 & (9) \\
(b_1, b_2^1, b_6^1) : \quad & x_1 + z_1 + y_2 + z_2 \geq y_3 + w_3 & (36) \\
(b_1, b_3^1, b_5^1) : \quad & x_1 + z_1 + y_2 + w_2 \geq z_3 + w_3 & (18) \\
(b_1, b_3^1, b_5^2) : \quad & x_1 + z_1 + y_2 + w_2 \geq y_3 - z_3 & (18) \\
(b_2^1, b_2^1, b_5^1) : \quad & y_1 + z_1 + y_2 + z_2 \geq z_3 + w_3 & (18) \\
(b_2^1, b_2^2, b_5^2) : \quad & y_1 + z_1 + x_2 + w_2 \geq y_3 - z_3 & (18)
\end{aligned}$$

WTI(2):

$$\begin{aligned}
(1, 1, b_9) : \quad & x_1 + y_1 + x_2 + y_2 - x_3 - y_3 \geq 0 & (3) \\
(b_1, 1, b_8) : \quad & x_1 + z_1 + x_2 + y_2 - x_3 - z_3 \geq 0 & (6) \\
(b_2^1, 1, b_7^1) : \quad & y_1 + z_1 + x_2 + y_2 - y_3 - z_3 \geq 0 & (18) \\
(b_3^1, 1, b_6^1) : \quad & y_1 + w_1 + x_2 + y_2 - y_3 - w_3 \geq 0 & (18) \\
(b_4^1, 1, b_5^1) : \quad & z_1 + w_1 + x_2 + y_2 - z_3 - w_3 \geq 0 & (18) \\
(b_4^2, 1, b_5^2) : \quad & y_1 - z_1 + x_2 + y_2 - y_3 + z_3 \geq 0 & (6)
\end{aligned}$$

The group $S_3 \times F$ acts canonically on \mathfrak{a}^3 , where S_3 acts by permutation of the three factors and F acts via the corresponding Dynkin automorphism of \mathfrak{a} . To get the full set of inequalities $T_{\mathbf{w}}^{P_2}$ for P_2 , we need to apply the group $S_3 \times F$ to the above collection of inequalities.

Thus, we get totally 186 inequalities corresponding to the maximal parabolic P_2 .

The multiplication table for $H^*(G/P_3)$ (resp. $H^*(G/P_4)$) can be obtained from that of $H^*(G/P_1)$ by using the isomorphism of $H^*(G/P_1)$ with $H^*(G/P_3)$ (resp. $H^*(G/P_4)$) induced from the Dynkin automorphisms. Accordingly, the inequalities corresponding to $H^*(G/P_3)$ and $H^*(G/P_4)$ are obtained from $T_{\mathbf{w}}^{P_1}$ by applying the action of F . All in all, each system $T_{\mathbf{w}}^{P_3}$ and $T_{\mathbf{w}}^{P_4}$ consists of 36 inequalities.

Below are the explicit lists of inequalities.

5.3. The system of inequalities corresponding to the parabolic subgroup P_3 .

ETI(3):

$$\begin{aligned}
x_1 + y_1 - z_1 + w_1 + x_2 + y_2 - z_2 + w_2 - x_3 + y_3 - z_3 - w_3 &\geq 0 \\
x_1 + y_1 - z_1 + w_1 + x_2 - y_2 + z_2 + w_2 - x_3 + y_3 + z_3 + w_3 &\geq 0 \\
x_1 - y_1 + z_1 + w_1 + x_2 - y_2 - z_2 - w_2 + x_3 + y_3 - z_3 + w_3 &\geq 0
\end{aligned}$$

WTI(3):

$$\begin{aligned}
x_1 + y_1 + z_1 - w_1 + x_2 + y_2 + z_2 - w_2 - x_3 - y_3 - z_3 + w_3 &\geq 0 \\
x_1 + y_1 + z_1 - w_1 + x_2 + y_2 - z_2 + w_2 - x_3 - y_3 + z_3 - w_3 &\geq 0 \\
x_1 + y_1 + z_1 - w_1 + x_2 - y_2 + z_2 + w_2 - x_3 + y_3 - z_3 - w_3 &\geq 0 \\
x_1 + y_1 + z_1 - w_1 - x_2 + y_2 + z_2 + w_2 + x_3 - y_3 - z_3 - w_3 &\geq 0
\end{aligned}$$

To get the full set of inequalities $T_{\mathbf{w}}^{P_3}$ for P_3 , we need to permute the above collection of inequalities where the subscripts $\{1, 2, 3\}$ are permuted arbitrarily.

5.4. The system of inequalities corresponding to the parabolic subgroup P_4 .

ETI(4):

$$\begin{aligned}
x_1 + y_1 - z_1 - w_1 + x_2 + y_2 - z_2 - w_2 - x_3 + y_3 - z_3 + w_3 &\geq 0 \\
x_1 + y_1 - z_1 - w_1 + x_2 - y_2 + z_2 - w_2 + x_3 - y_3 - z_3 + w_3 &\geq 0 \\
x_1 - y_1 + z_1 - w_1 - x_2 + y_2 + z_2 - w_2 + x_3 + y_3 - z_3 - w_3 &\geq 0
\end{aligned}$$

WTI(4):

$$\begin{aligned}
x_1 + y_1 + z_1 + w_1 + x_2 + y_2 + z_2 + w_2 - x_3 - y_3 - z_3 - w_3 &\geq 0 \\
x_1 + y_1 + z_1 + w_1 + x_2 + y_2 - z_2 - w_2 - x_3 - y_3 + z_3 + w_3 &\geq 0 \\
x_1 + y_1 + z_1 + w_1 + x_2 - y_2 + z_2 - w_2 - x_3 + y_3 - z_3 + w_3 &\geq 0 \\
x_1 + y_1 + z_1 + w_1 + x_2 - y_2 - z_2 + w_2 - x_3 + y_3 + z_3 - w_3 &\geq 0
\end{aligned}$$

To get the full set of inequalities $T_{\mathbf{w}}^{P_4}$ for P_4 , we need to permute the above collection of inequalities where the subscripts $\{1, 2, 3\}$ are permuted arbitrarily.

5.5. The cone \mathcal{C} . Thus, the total number of inequalities $T_{\mathbf{w}}^{P_i}$ defining the cone \mathcal{C} inside Δ^3 is equal to $36 + 186 + 36 + 36 = 294$. Since $\Delta^3 \subset \mathfrak{a}^3$ is defined by 12 inequalities, we get altogether 306 inequalities defining the cone \mathcal{C} inside \mathfrak{a}^3 . Let Σ be the set of these 306 inequalities defining the cone \mathcal{C} .

Theorem 5.1. *The system Σ is irredundant.*

Proof. In order to show the irredundancy of the system Σ , it suffices to show that the cone \mathcal{C} has 306 faces. It is done by applying the MAPLE package CONVEX [F] to the above system (see [Ka1]). \square

Remark 5.2. *The same computation also shows that the cone \mathcal{C} has 81 extremal rays.*

Our next goal is to verify the *saturation conjecture* for the group $G^\vee = \text{Spin}(8)$. Let $P^\vee \subset (\mathfrak{h}^\vee)^* = \mathfrak{h}$ denote the weight lattice of G^\vee and $Q^\vee \subset P^\vee$ denote the root lattice. Of course, $Q = Q^\vee$ since G is simply-laced. Recall that in the Introduction we have defined the semigroups \mathcal{C}^0 and \mathcal{R} of \mathcal{D}^3 with

$$\mathcal{R} \subset \mathcal{C}^0.$$

Theorem 5.3 (Saturation theorem for $\text{Spin}(8)$).

$$\mathcal{R} = \mathcal{C}^0.$$

Proof. In order to prove the inclusion $\mathcal{C}^0 \subset \mathcal{R}$, it suffices to show that each semigroup generator of \mathcal{C}^0 belongs to \mathcal{R} . To find the minimal set of semigroup generators (Hilbert basis) for \mathcal{C}^0 , we define a basis $\{\bar{\alpha}_i, \zeta_j\}_{1 \leq i \leq 4, 1 \leq j \leq 8}$ of the lattice $\phi^{-1}(Q^\vee)$, where

$$\phi : (P^\vee)^3 \rightarrow P^\vee, \quad \phi(\lambda, \mu, \nu) = \lambda + \mu + \nu.$$

Consider the splitting of the exact sequence (for $K := \text{Ker } \phi$)

$$0 \rightarrow K \rightarrow (P^\vee)^3 \xrightarrow{\phi} P^\vee \rightarrow 0$$

over Q^\vee under the map $\psi(\alpha_i) = (\alpha_i, 0, 0), i = 1, \dots, 4$.

Therefore, we can identify Q^\vee with the subgroup $\psi(Q^\vee) \subset (P^\vee)^3$ with basis

$$\{\bar{\alpha}_i = \psi(\alpha_i)\}_{1 \leq i \leq 4}.$$

Set

$$\zeta_j = (-\omega_j, \omega_j, 0)$$

for $1 \leq j \leq 4$ and

$$\zeta_j = (-\omega_{j-4}, 0, \omega_{j-4})$$

for $5 \leq j \leq 8$, where $\{\omega_j\}$ are the fundamental weights for $\text{Spin}(8)$. Then, it is clear that $\{\zeta_j, j = 1, \dots, 8\}$ is a basis of K and

$$\{\bar{\alpha}_i, \zeta_j : i = 1, \dots, 4, j = 1, \dots, 8\}$$

is a basis of $\phi^{-1}(Q^\vee)$.

Thus, the semigroup \mathcal{C}^0 is precisely equal to the integral points of the cone \mathcal{C} with respect to the coordinates $\{\bar{\alpha}_i, \zeta_j\}$.

Computation of the Hilbert basis H of \mathcal{C}^0 is done via the package HILBERT, [HHM] (see [Ka2]). Observe that the action of the group $S_3 \times F$ on \mathfrak{a}^3 keeps \mathcal{C}^0 stable. Since the Hilbert basis is unique, it

follows that H is invariant under the action of $S_3 \times F$. Below is the list H' of elements of H modulo the action of $S_3 \times F$:

$$\begin{aligned} &(\omega_1, \omega_1, 0) \\ &(\omega_2, \omega_2, 0) \\ &(\omega_2, \omega_2, \omega_2) \\ &(\omega_1, \omega_3, \omega_4) \\ &(\omega_1, \omega_1, \omega_2) \\ &(\omega_1, \omega_2, \omega_3 + \omega_4) \\ &(2\omega_1, \omega_2, \omega_2) \\ &(\omega_1 + \omega_2, \omega_2, \omega_3 + \omega_4) \\ &(\omega_2, \omega_2, \omega_1 + \omega_3 + \omega_4) \\ &(2\omega_2, \omega_2, \omega_1 + \omega_3 + \omega_4). \end{aligned}$$

Since $S_3 \times F$ also preserves the semigroup \mathcal{R} , in order to prove Theorem 5.3, it suffices to check that $H' \subset \mathcal{R}$. This is done using MAPLE package WEYL written by John Stembridge, see [S]. It is done in [Ka2].

REFERENCES

- [AMW] A. Alekseev, E. Meinrenken and C. Woodward, *Linearization of Poisson actions and singular values of matrix products*, Ann. Inst. Four., vol. **51** (2001), 1697–1717.
- [B] P. Belkale, *Geometric proofs of Horn and saturation conjectures*, J. Alg. Geom., vol. **15** (2006), 133–173.
- [BK] P. Belkale and S. Kumar, *Eigenvalue problem and a new product in cohomology of flag varieties*, Inventiones Math. (2006).
- [BS] A. Berenstein and R. Sjamaar, *Projections of coadjoint orbits, moment polytopes and the Hilbert–Mumford criterion*, J. Amer. Math. Soc., vol. **13** (2000), 433–466.
- [Bo] N. Bourbaki, “Groupes et Algèbres de Lie”, Chap. 4, 5, 6, Hermann, 1981.
- [DW] H. Derksen and J. Weyman, *Semi-invariants of quivers and saturation for Littlewood–Richardson coefficients*, J. Amer. Math. Soc., vol. **13** (2000), 467–479.
- [EL] S. Evens and J.-H. Lu, *Thompson’s conjecture for real semisimple Lie groups*, In: “The breadth of symplectic and Poisson geometry,” Festschrift in honor of Alan Weinstein, Birkhäuser, Progress in Math., vol. **232** (2005), 121–172.
- [F] M. Franz, *Convex — a Maple package for convex geometry*, available at <http://www-fourier.ujf-grenoble.fr/~franz/convex>, 2005.
- [He] S. Helgason, “Differential Geometry, Lie groups, and Symmetric Spaces”, Academic Press, 1978.
- [HHM] R. Hemmecke, R. Hemmecke and P. Malkin, *4ti2, Version 1.2—Computation of Hilbert bases, Graver bases, toric Gröbner bases, and more*, Available at www.4ti2.de, 2005.

- [Ka1] M. Kapovich, *Proof of irredundancy of Belkale–Kumar stability inequalities for $PO(8)$* . Available at <http://www.math.ucdavis.edu/~kapovich/EPR/irred.mw>
- [Ka2] M. Kapovich, *Proof of Saturation Theorem for the tensor products of irreducible representations of $Spin(8)$* . Available at <http://www.math.ucdavis.edu/~kapovich/EPR/hbasis.mw>
- [KLM1] M. Kapovich, B. Leeb and J. J. Millson, *Convex functions on symmetric spaces, side lengths of polygons and the stability inequalities for weighted configurations at infinity*, Preprint, 2005, arXiv:math.DG/0311486.
- [KLM2] M. Kapovich, B. Leeb, J. J. Millson, *Polygons in buildings and their refined side-lengths*, Preprint, 2004, arXiv:math.MG/0406305.
- [KLM3] M. Kapovich, B. Leeb, J. J. Millson, *The generalized triangle inequalities in symmetric spaces and buildings with applications to algebra*, Preprint, 2004, arXiv:math.RT/0210256, to appear in the Memoirs of the AMS.
- [KM1] M. Kapovich and J. J. Millson, *A path model for geodesics in Euclidean buildings and its applications to representation theory*, Preprint, 2004, arXiv:math.RT/0411182.
- [KM2] M. Kapovich and J. J. Millson, *Structure of the tensor product semigroup*, Preprint, 2005, arXiv:math.RT/0508186.
- [Kl1] A. Klyachko, *Stable bundles, representation theory and Hermitian operators*, Selecta Mathematica, vol. **4** (1998), 419–445.
- [Kl2] A. Klyachko, *Random walks on symmetric spaces and inequalities for matrix*, Linear Algebra Appl. vol. **319** (2000), 37–59.
- [KT] A. Knutson and T. Tao, *The honeycomb model of $GL(n)$ tensor products I: Proof of the saturation conjecture*, J. Amer. Math. Soc., vol. **12** (1999), 1055–1090.
- [KTW] A. Knutson, T. Tao, C. Woodward, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products II: Puzzles determine the facets of the Littlewood–Richardson cone*, J. Amer. Math. Soc., vol. **17** (2004), 19–48.
- [K] S. Kumar, “Kac–Moody groups, their flag varieties and representation theory”, Birkhauser, Progress in Mathematics, vol. **204**, 2002.
- [KuLM] S. Kumar, B. Leeb and J. J. Millson, *The generalized triangle inequalities for rank 3 symmetric spaces of noncompact type*, Contemp. Math., vol. **332** (2003), 171–195.
- [Sp] T. A. Springer, *Reductive groups*, In: “Automorphic forms, representations and L-functions,” Proc. Symp. AMS, vol. **33** (1979), 3–27.
- [S] J. Stembridge, *Coxeter and Weyl Maple packages*, Available at www.math.lsa.umich.edu/~jrs/maple.html

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